Isop. Profile

1) Introduction

1936 Sobolev

$$\|f\|_{\frac{np}{n-p}} \leqslant C \|\nabla f\|_p \qquad 1 \leqslant p < n$$

for all $f \in C_c^\infty(\mathbb{R}^n)$
some constant $C>0$

$p=1,\ n=2$

$$|f(x_0, y_0)| \leqslant \int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial x}(x, y_0)\right| dx$$

$$|f(x_0,y_0)| \leqslant \int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial y}(x_0, y)\right| dy$$

$$|f(x_0,y_0)|^2 \leqslant \int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial x}(x,y_0)\right| dx \int_{-\infty}^{+\infty} \left|\frac{\partial f}{\partial y}(x_0,y)\right| dy$$

$$\int |f(x_0, y_0)|^2 dx_0 dy_0 \leqslant \left\|\frac{\partial f}{\partial x}\right\|_1 \left\|\frac{\partial f}{\partial y}\right\|_1$$

$$\|f\|_2^2 \leqslant \left\|\frac{\partial f}{\partial x}\right\|_1 \left\|\frac{\partial f}{\partial y}\right\|_1$$

$$\|f\|_2 \leqslant \frac{1}{2}\left(\|\nabla f\|_1\right)$$

Ω finite measure space

$$f \in L^p(\Omega) \Rightarrow f \in L^q(\Omega) \qquad q < p$$

$\underline{K>1}$ $\quad \|f\|_p \leq \|f\|_{pK}\, \mu(\operatorname{supp} f)^{\frac{K-1}{Kp}}$

$$\|f\|_{\frac{np}{n-p}} \leq C\|\nabla f\|_p$$

$$\|f\|_p \leq \|f\|_{\frac{np}{n-p}}\, \mu(\operatorname{supp} f)^{\frac{1}{n}}$$

$$\|f\|_p \leq C\|\nabla f\|_p\, \mu(\operatorname{supp} f)^{\frac{1}{n}}$$

Kanai 85 Large scale Sobolev inequalities

(M,g) + bounded geometry $\quad \begin{array}{c}\tilde{M} \\ \downarrow \\ M\end{array}$

M compact

to sobolev inequalities on the discretization

2) Isop. Profile on graphs

$\Gamma = (V,E)$ graph

$A \subset V$

$\partial A = \{ e = \{x,y\} \mid x \in A, y \notin A \}$

$c_0(V) = \{$ fin. supported functions on $V \}$

$f \in C_0(V)$

$$\|\nabla f\|_p^p = \sum_{e=\{x,y\} \in E} |f(x) - f(y)|^p$$

$$J_{p,\Gamma}(n) = \sup_{|\mathrm{supp} f| \leq n} \frac{\|f\|_p}{\|\nabla f\|_p}$$

Γ has bounded valency

$\Gamma = \langle S \rangle$, S finite

$\Gamma = \mathrm{Cay}(\Gamma, S)$ $\quad\quad$ $\Gamma = \langle S \rangle = \langle T \rangle$

$$J_{p,\Gamma,S} \approx J_{p,\Gamma,T}$$

$$\|\nabla_S f\|_p^p = \sum_{\gamma \in \Gamma, s \in S} |f(\gamma) - f(\gamma s)|^p$$

$$= \sum_{s \in S} \|f - \rho(s) f\|_p^p$$

ρ right regular representation

$\Gamma = \langle S \rangle = \langle T \rangle$ $\quad\quad$ $T \subset S^N$ $\quad\quad$ $N > 0$

$$\|\nabla_T f\|_p \leqslant \|\nabla_{S^N} f\|_p$$

$$\|\nabla_{S^N} f\|_p^p \leqslant |S^N|\, N^p \|\nabla_S f\|_p^p \quad \circ\circ\circ\circ \quad (A)$$

$$\|$$

$$\sum_{t \in S^N} \|f - \rho(t) f\|_p^p$$

$$\|f - \rho(t) f\|_p^p \leqslant N^p \|\nabla_S f\|_p^p$$

$t = s_1 s_2 \ldots, s_N$ $\quad$ $s_i \in S$

$$\|f-\rho(t)f\|_p \leq \|f-\rho(s_1)f\|_p + \|\rho(s_1s_2)f - \rho(s_1 f)\|_p$$
$$+ \|\rho(s_1s_2s_3)f - \rho(s_1s_2)f\|_p + \dots$$
$$\|\rho(s_1\dots s_{N-1})f - \rho(s_1\dots s_N)f\|_p$$
$$= \sum_{i=1}^{N} \|f-\rho(s_i)f\|_p$$
$$\leq N\|\nabla_S f\|_p$$

$$J_{1,\Gamma}(n) \approx \sup_{|A|\leq n} \frac{|A|}{|\partial A|}$$

$f \approx g \Leftrightarrow f \preceq g,\ g \preceq f,$ $\qquad f \preceq g,\ \ f(n) \leq C\, g(Cn)$

3) Q.I-Invariance

$\varphi : \Gamma \to \Lambda$ $\qquad$ Q.I

(Γ, S_Γ) $\qquad \Downarrow \qquad \varphi(\Gamma)$ R-dense

(Λ, S_Λ) $\qquad \tilde{\varphi} : \Gamma \times B_\Lambda(1,R) \to \Lambda$

$\qquad (\gamma, t) \mapsto \varphi(\gamma)t$

φ is surjective

$$f \in C_0(\Lambda) \Rightarrow f \circ \varphi \in C_0(\Gamma)$$

$\exists M$ s.t

$\forall \lambda \in \Lambda, \quad |\varphi^{-1}(\lambda)| \leq M.$

$\exists L$ s.t. $\qquad d(\varphi(\gamma), \varphi(\gamma s)) \leq L$

$\qquad \forall \gamma \in \Gamma, s \in S_\Gamma$

$$\|f\|_p \leq \|f\circ\varphi\|_p \leq M\|f\|_p$$

φ surjective $\qquad |\varphi^{-1}(\lambda)| \leq M$

$$\|\nabla_{S_\Gamma}(f\circ\varphi)\|_p^p = \sum_{\gamma\in\Gamma, s\in S_\Gamma} |f(\varphi(\gamma)) - f(\varphi(\gamma s))|^p$$

$$\leqslant |S_\Gamma| \sum_{\gamma\in\Gamma, t\in S_\Lambda^L} |f(\varphi(\gamma)) - f(\varphi(\gamma)t)|^p$$

$$\leqslant M|S_\Gamma| \sum_{\lambda\in\Lambda, t\in S_\Lambda^L} |f(\lambda) - f(\lambda t)|^p$$

$$\leqslant M|S_\Gamma| \, \|\nabla_{S_\Lambda^L} f\|_p^p$$

So we have proved the following

$$\|f\|_p \leqslant \|f\circ\varphi\|_p \leqslant M\|f\|_p$$

$$\|\nabla f\circ\varphi\|_p \leqslant C\|\nabla f\|_p$$

$$J_{1,\Lambda} \preceq J_{1,\Gamma}$$

4) Links with Growth: Coulhon-Saloff-Coste Inequality

$X := (V, E)$ graph

We say X is quasi-homogeneous (or it satisfies a pseudo Poincaré inequality) if

$$\|f - P_n f\|_1 \leqslant \eta\, n \|\nabla f\|_1$$

for all $n\in\mathbb{N}$, $f\in C_0(V)$

$$P_n f(y) = \frac{1}{|B(y,n)|} \sum_{z\in B(y,n)} f(z)$$

If (X, E) satisfies

$$|A|^{1-1/d} \leqslant C_1 |\partial A| \quad \forall A$$

we can put $A_n = B(x,n)$

$$|A_n|^{1-1/d} \leqslant C_1 |A_{n+1} - A_n|$$

$$|A_n| \succeq n^d$$

growth information + quasi-homogeneity

$\Downarrow$

information about J_1, Γ

quasi-homogeneity $\Leftarrow$ Cayley graphs

$$|S^N| . P_n f(\cancel{\gamma}) = \sum_{t \in S^N} f(\gamma t)$$

$$\|f - P_n f\|_1$$

$$\wedge\!\!\!\!/\!/$$

$$\frac{1}{|S^N|} \sum_{\gamma \in \Gamma, t \in S^N} |f(\gamma) - f(\gamma t)|$$

$$\leqslant N \|\nabla_S f\|_1$$

<u>PROP</u>: (C-S.C)

X quasi-homogeneous.

$$v(n) = \inf_{x \in X} |B(x,n)|$$

$$\varphi(t) = \min\{ n : v(n) > t \}$$

$$\psi(t) = \frac{t}{\varphi(2t)}$$

Isop. Ineq. relating ψ

$A \subset X$, finite

$m(A) = |A|$ $\qquad\qquad \psi(|A|) \leq C\,|\partial A|$

$$m(A) \leq m\left(|1_A - P_n 1_A| \geq \tfrac{1}{2}\right) + m\left(P_n 1_A \geq \tfrac{1}{2}\right)$$

$$\begin{array}{c} m(A) \\ \| \\ m(1_A) \end{array}$$

$$\begin{aligned} m\left(|1_A - P_n 1_A| \geq \tfrac{1}{2}\right) &\leq 2\,\|1_A - P_n 1_A\|_1 \\ &\leq 2 n \eta \,\|\nabla 1_A\|_1 \\ &\leq 2 n \eta\, |\partial A| \end{aligned}$$

$n = \varphi(2|A|)$ $\qquad \varphi(t) = \min\{\, n \mid V(n) > t \,\}$

$v(n) > 2|A|$

$|B(x,n)| > 2|A| \quad \forall\, x \in X$

$$P_n 1_A(x) \leq \frac{|A|}{B(x,n)} < \frac{1}{2}$$

$$|A| \leq 2\varphi(2|A|)\,\eta\,|\partial A|$$

Γ Cayley graph $\qquad v_\Gamma(n) \succeq n^d \qquad e^{n^c}$

$$\Downarrow$$

$$J_{1,\Gamma}(n) \preceq n^{1/d} \qquad -$$

$V_\Gamma(n) \simeq e^n$ $\qquad J_{1,\Gamma}(n) \preceq \log n$

$V_\Gamma(n) \approx e^{n^\alpha}$ $\qquad J_{1,\Gamma}(n) \preceq (\log n)^{1/\alpha}$

$0 < \alpha < 1$